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Category-theoretic aspects of abstract elementary classes

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ABSTRACT

We highlight connections between accessible categories and abstract elementary classes (AECs), and provide a dictionary for translating properties and results between the two contexts. We also illustrate a few applications of purely category-theoretic methods to the study of AECs, with model-theoretically novel results. In particular, the category-theoretic approach yields two surprising consequences: a structure theorem for categorical AECs, and a partial stability spectrum for weakly tame AECs.

1. Introduction

There has been a recent surge of interest in the model theory of nonelementary classes, spurred by the appearance of a number of important applications in mainstream mathematics, including recent contributions to the understanding of Banach spaces and the field of complex numbers with exponentiation (the former arising in work of Itai Ben Yaakov, C. Ward Henson, and Jose Iovino, and the latter being the particular province of Boris Zilber, beginning in [16]). Two frameworks for the analysis of such classes recommend themselves due to the balance they strike between generality and richness of structure: abstract elementary classes (AECs) and accessible categories. Although these notions were generated in the course of independent lines of investigation – in model theory and categorical logic, respectively – they exhibit striking similarities. AECs, on the one hand, were introduced by Shelah as a broad framework in which to carry out the project of classification theory for a wide array of nonelementary classes. In contrast with earlier work on, say, the model theory of $L_{\omega_1,\omega}$, where the methods were closely tied to the structure of the ambient logic, in AECs one dispenses with syntax, retaining only the fundamental category-theoretic structure carried by the strong embeddings. Accessible categories, on the other hand, may be regarded as an outgrowth of categorical logic, the program in which logical theories are associated with categories that capture their essential structure and classical models are identified with structure-preserving **Set**-valued functors on the associated categories. In [1,5,12] one sees, in parallel with the story for AECs, a shift in emphasis away from the category associated with a theory, and a focus on the abstract properties of the category of models itself—in this way, one arrives at the notion of an accessible category.

The goal of the present inquiry (alongside [4], comprising independent work of Beke and Rosický) is to begin to fill in the details of the connection between AECs and accessible categories and to illustrate a few ways in which results from the world

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of accessible categories can be translated into novel results for AECs. As this paper represents an attempt at a rapprochement between model-theoretic and category-theoretic perspectives, we have endeavored to provide enough background detail in Sections 2 and 3 to accommodate readers whose experience tends to place them squarely on one side or the other of the divide. In Section 4, we begin the process of reconciliation, realizing AECs as highly structured accessible categories, and giving a complete category-theoretic axiomatization of AECs in a finitary signature L as subcategories of the ambient category of L-structures. Section 5 translates a number of notions from accessible categories (most of which are drawn from [13]) into the context of AECs. This exercise bears immediate fruit in Sections 6 and 7, as simple category-theoretic manipulations yield a pair of novel results: respectively, a structure theorem for categorical AECs, and a partial stability spectrum result for weakly tame AECs that are totally transcendental in the sense of [11].

2. Abstract elementary classes

We begin with a very brief introduction to AECs, Galois types, and a few relevant properties thereof. Readers interested in further details may wish to consult [2] or [6]. To begin:

Definition 2.1. Let L be a finitary signature (one-sorted, for simplicity). A class of L-structures $\mathcal{K}$ equipped with a partial order $\prec_{\mathcal{K}}$ is an *abstract elementary class (AEC)* if it satisfies the following axioms:

A0 (Closure under isomorphism)
  (i) For every $M \in \mathcal{K}$ and L-structure N with $N \cong M$, $N \in \mathcal{K}$.
  (ii) If $N_1, N_2, M_1, M_2 \in \mathcal{K}$ and there exist isomorphisms $f_i : N_i \cong M_i$ (for $i = 1, 2$) with $f_1 \subseteq f_2$, then $N_1 \prec_{\mathcal{K}} N_2$ implies $M_1 \prec_{\mathcal{K}} M_2$.
A1 For all M, N in $\mathcal{K}$, if $M \prec_{\mathcal{K}} N$, then $M \subseteq_L N$.
A2 (Unions of Chains) Let $(M_\alpha | \alpha < \delta)$ be a continuous $\prec_{\mathcal{K}}$-increasing sequence.
  1. $\bigcup_{\alpha<\delta} M_\alpha \in \mathcal{K}$.
  2. For all $\beta < \delta$, $M_\beta \prec_{\mathcal{K}} \bigcup_{\alpha<\delta} M_\alpha$.
  3. If $M_\alpha \prec_{\mathcal{K}} M$ for all $\alpha < \delta$, then $\bigcup_{\alpha<\delta} M_\alpha \prec_{\mathcal{K}} M$.
A3 (Coherence) If $M_0, M_1 \prec_{\mathcal{K}} M$ in $\mathcal{K}$, and $M_0 \subseteq_L M_1$, then $M_0 \prec_{\mathcal{K}} M_1$.
A4 (Downward Löwenheim–Skolem) There exists an infinite cardinal $\mathrm{LS}(\mathcal{K})$ with the property that for any $M \in \mathcal{K}$ and subset A of M, there exists $M_0 \in \mathcal{K}$ with $A \subseteq M_0 \prec_{\mathcal{K}} M$ and $|M_0| \leq |A| + \mathrm{LS}(\mathcal{K})$.

The prototypical example, of course, is the case in which $\mathcal{K}$ is an elementary class – the class of models of a particular first order theory T – and $\prec_{\mathcal{K}}$ is the elementary submodel relation, in which case $\mathrm{LS}(\mathcal{K})$ is, naturally, $\aleph_0 + |L(T)|$.

For any infinite cardinal λ, we denote by $\mathcal{K}_\lambda$ the subclass of $\mathcal{K}$ consisting of all models of cardinality λ (with the obvious interpretations for such notations as $\mathcal{K}_{\leq\lambda}$ and $\mathcal{K}_{>\lambda}$). We say that $\mathcal{K}$ is λ-categorical if $\mathcal{K}_\lambda$ contains only a single model up to isomorphism. For $M, N \in \mathcal{K}$, we say that a map $f : M \to N$ is a $\mathcal{K}$-embedding (or, more often, a strong embedding) if f is an injective homomorphism of $L(\mathcal{K})$-structures, and $f[M] \prec_{\mathcal{K}} N$; that is, f induces an isomorphism of M onto a strong submodel of N. In that case, we write $f : M \hookrightarrow_{\mathcal{K}} N$.

Definition 2.2. We say that an AEC $\mathcal{K}$ has the *joint embedding property (JEP)* if for any $M_1, M_2 \in \mathcal{K}$, there is an $M \in \mathcal{K}$ that admits strong embeddings of both M_1 and M_2.

Definition 2.3. We say that an AEC $\mathcal{K}$ has the *amalgamation property (AP)* if for any $M_0 \in \mathcal{K}$ and strong embeddings $f_1 : M_0 \hookrightarrow_{\mathcal{K}} M_1$ and $f_2 : M_0 \hookrightarrow_{\mathcal{K}} M_2$, there are strong embeddings $g_1 : M_1 \hookrightarrow_{\mathcal{K}} N$ and $g_2 : M_2 \hookrightarrow_{\mathcal{K}} N$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

It is not immediately clear what we might embrace as a suitable notion of type in AECs, given that we have dispensed with syntax, and removed ourselves to a world of abstract embeddings and diagrams thereof. The best candidate – the Galois type – has its origins in the work of Shelah (see [14]). In the most general formulation, a Galois type is an equivalence class of triples of the form (M, a, N) with $M \prec_{\mathcal{K}} N$ and $a \in N$ under a relation $\sim$ defined as follows: $(M, a_1, N_1) \sim (M, a_1, N_1)$ if there is a model N and a pair of embeddings $f_1 : N_1 \hookrightarrow_{\mathcal{K}} N$ and $f_2 : N_2 \hookrightarrow_{\mathcal{K}} N$ such that $f_1 \restriction M = f_2 \restriction M$ – that is, such that the following diagram commutes

$$\begin{tikzcd}
N_1 \arrow[r, "f_1"] & N \\
M \arrow[u] \arrow[r] & N_2 \arrow[u, "f_2"']
\end{tikzcd}$$

– and, moreover, $f_1(a_1) = f_2(a_2)$. While the relation $\sim$ is automatically reflexive and symmetric, transitivity follows from the amalgamation property (Remark I.1.2 in [15]). Indeed, for the purposes of our discussion of Galois types in this section and in Section 7, we will assume both amalgamation and joint embedding. Under these twin assumptions, one can show that the AEC contains a large strongly model homogeneous structure $\mathfrak{C}$ – roughly speaking, for every $M \in \mathcal{K}$ with $|M| < |\mathfrak{C}|$,

$M \prec_{\mathcal{K}} \mathfrak{C}$, and any strong embedding $f : M \hookrightarrow_{\mathcal{K}} M'$ with $|M|, |M'| < |\mathfrak{C}|$ extends to an automorphism of $\mathfrak{C}$ – which is referred to as the monster model and shares all the properties of its first order analogues (see the discussion following Theorem 8.5 in [2]). In this case, the definition of Galois types reduces to the following equivalent (but far simpler) characterization:

Definition 2.4. Let $M \in \mathcal{K}$, and $a \in \mathfrak{C}$. The *Galois type of a over M*, denoted ga-tp(a/M), is the orbit of a in $\mathfrak{C}$ under $\mathrm{Aut}_M(\mathfrak{C})$, the group of automorphisms of $\mathfrak{C}$ that fix M. We denote by ga-S(M) the set of all Galois types over M.

In case $\mathcal{K}$ is an elementary class with $\prec_{\mathcal{K}}$ as elementary submodel, the Galois types over M correspond to the complete first order types over M:

$$\text{ga-tp}(a/M) = \text{ga-tp}(b/M) \text{ if and only if } \text{tp}(a/M) = \text{tp}(b/M).$$

In general, however, Galois types and syntactic types do not match up, even in cases when the logic underlying the AEC is clear (say, $\mathcal{K} = \mathrm{Mod}(\psi)$, with $\psi \in L_{\omega_1,\omega}$). A few basic definitions and notations:

Definition 2.5. 1. We say that $\mathcal{K}$ is *λ-Galois stable* if for every $M \in \mathcal{K}_\lambda$, $|\text{ga-S}(M)| = \lambda$.
2. For any M, $a \in \mathfrak{C}$, and $N \prec_{\mathcal{K}} M$, the *restriction of ga-tp(a/M) to N*, which we denote by ga-tp$(a/M) \upharpoonright N$, is the orbit of a under $\mathrm{Aut}_N(\mathfrak{C})$.
3. Let $N \prec_{\mathcal{K}} M$ and $p \in$ ga-S(N). We say that M *realizes* p if there is an element $a \in M$ such that ga-tp$(a/M) \upharpoonright N = p$. Equivalently, M realizes p if the orbit in $\mathfrak{C}$ corresponding to p meets M.
4. We say that a model M is *λ-Galois-saturated* if for every $N \prec_{\mathcal{K}} M$ with $|N| < \lambda$ and every $p \in$ ga-S(N), p is realized in M. We say that M is Galois-saturated if it is $|M|$-Galois-saturated.

Henceforth, the word "type" should be understood to mean "Galois type", unless otherwise indicated. It bears mentioning that Galois saturation is closely related to a notion of homogeneity peculiar to AECs:

Definition 2.6. A model $M \in \mathcal{K}$ is *λ-model homogeneous* if for any $N \prec_{\mathcal{K}} M$ and $N' \in \mathcal{K}_{<\lambda}$ with $N \prec_{\mathcal{K}} N'$, there is an embedding of N' into M that fixes N. We say M is model homogeneous if it is $|M|$-model homogeneous.

It is worth noting that the monster model $\mathfrak{C}$ introduced above is model homogeneous in this sense. We will have occasion to use the following fact (Theorem 8.14 in [2]):

Proposition 2.7. *Let $\mathcal{K}$ be an AEC satisfying AP and JEP. For $\lambda > LS(\mathcal{K})$, a model $M \in \mathcal{K}$ is λ-Galois-saturated if and only if it is λ-model homogeneous.*

Initial attempts at establishing a classification theory for AECs have focused on classes satisfying a variety of broad structural conditions. We here concern ourselves primarily with AECs satisfying the property known as tameness, which says, roughly speaking, that types are determined by their restrictions to small submodels of their domains, a condition reminiscent of the locality properties of syntactic types.

Definition 2.8. We say that $\mathcal{K}$ is *χ-tame* if for every $M \in \mathcal{K}$, if p and p' are distinct types over M, then there is an $N \prec_{\mathcal{K}} M$ with $|N| \leq \chi$ such that $p \upharpoonright N \neq p' \upharpoonright N$. We say that $\mathcal{K}$ is *weakly χ-tame* if the condition above holds for Galois-saturated $M \in \mathcal{K}$.

Informally, we say that an AEC is "tame" or "weakly tame" if it is, respectively, χ-tame or weakly χ-tame for some cardinal χ. Tameness plays a crucial role in existing results on the classification theory of AECs (a field which remains, it must be said, a work in progress). In regard to questions of eventual categoricity, results are typically measured against Shelah's Categoricity Conjecture: if an AEC $\mathcal{K}$ is categorical in one cardinal $\mu \geq \mathrm{Hanf}(\mathcal{K})$, the Hanf number of the class (see [2] for a detailed treatment of this notion), $\mathcal{K}$ is categorical in every $\kappa \geq \mathrm{Hanf}(\mathcal{K})$. Approximations of this result hold in tame AECs, the most promising of which, a result of [8], implies categoricity in every cardinal $\kappa \geq \mathrm{H}_2$, the second Hanf number—which is related to, but substantially larger than, $\mathrm{Hanf}(\mathcal{K})$—given categoricity in a single successor cardinal $\mu^+ \geq \mathrm{H}_2$. It is important to note that the computation of the Hanf number requires the reintroduction of syntax via Shelah's Presentation Theorem, and the proof of the eventual categoricity result mentioned above depends on syntax and a resort to a classical, and decidedly set-theoretic, toolkit: indiscernibles, EM-models, and so on. A natural question: if we retain our category-theoretic perspective, can we still prove interesting theorems about categorical AECs? Section 6 represents a partial answer (in the affirmative) and suggests that there are results which, although readily apparent from that perspective, would be otherwise unobtainable.

Stability spectrum results are even patchier. For tame AECs, Grossberg and VanDieren have proven in [7], using splitting and the techniques mentioned above, that stability in a cardinal λ implies stability in any κ such that $\kappa^\lambda = \kappa$. In [3], the machinery of splitting is invoked again, this time to prove stability transfer from a cardinal λ to λ^+, a result that carries over to the weakly tame context. In [11], the author introduces a few new tools for the analysis of Galois stability in AECs, chiefly a family of Morley-like ranks RM^λ, indexed by cardinals $\lambda \geq \mathrm{LS}(\mathcal{K})$, which, although there considered only in the tame and weakly tame contexts, make sense in any AEC with amalgamation and joint embedding (although the so-called Quasi-unique Extension Property – Proposition 3.10 in [11] – may fail). Motivated by the intuition that types over small models are the analogues of formulas from the classical theory, and that the restrictions of a type to small submodels of its domain are, in a sense, its constituent types, one defines:

Definition 2.9. For $\lambda \geq \mathrm{LS}(\mathcal{K})$, we define RM^λ by the following induction: for any $q \in \text{ga-S}(M)$ with $|M| \leq \lambda$,

- $\mathrm{RM}^\lambda[q] \geq 0$.
- $\mathrm{RM}^\lambda[q] \geq \alpha$ for limit α if $\mathrm{RM}^\lambda[q] \geq \beta$ for all $\beta < \alpha$.
- $\mathrm{RM}^\lambda[q] \geq \alpha + 1$ if there exists a structure $M' \succ_{\mathcal{K}} M$ such that q has strictly more than λ extensions to types q' over M' with the property that

$$\mathrm{RM}^\lambda[q' \restriction N] \geq \alpha \text{ for all } N \in \mathrm{Sub}_{\leq\lambda}(M').$$

For types q over M of arbitrary size, we define

$$\mathrm{RM}^\lambda[q] = \min\{\mathrm{RM}^\lambda[q \restriction N] : N \in \mathrm{Sub}_{\leq\lambda}(M)\}.$$

As usual, if $\mathrm{RM}^\lambda[q] > \alpha$ for all ordinals α, we say that $\mathrm{RM}^\lambda[q] = \infty$. These ranks are suitably monotonic, invariant under automorphisms of $\mathfrak{C}$, and decreasing in the cardinal parameter λ (see Propositions 3.3, 3.4, and 3.6 in [11]). They also support new notions of total transcendence:

Definition 2.10. We say that a class is λ-*totally transcendental* if for every $M \in \mathcal{K}$ and $q \in \text{ga-S}(M)$, $\mathrm{RM}^\lambda[q]$ is an ordinal.

Since λ-total transcendence allows one to bound the number of types over structures in $\mathcal{K}_{>\lambda}$ and since, at least in tame AECs, λ-total transcendence follows from λ-stability provided $\lambda^{\aleph_0} > \lambda$, this allows one to prove a number of upward stability transfer theorems (see [11]). For example, it is shown that for any $\aleph_0$-tame and $\aleph_0$-Galois stable AEC $\mathcal{K}$, if $\mathcal{K}$ is Galois stable in a sequence of cardinals cofinal in a cardinal κ, $cf(\kappa) > \aleph_0$, then it is κ-Galois stable as well, generalizing a result of [3].

For weakly tame AECs, little is known beyond the transfer of stability from a cardinal to its successor, mentioned above, and the following result of [11]:

Theorem 2.11. *Let $\mathcal{K}$ be weakly χ-tame for some $\chi \geq LS(\mathcal{K})$, and μ-totally transcendental with $\mu \geq \chi$. Suppose that λ is a cardinal with $cf(\lambda) > \mu$, and that every $M \in \mathcal{K}_\lambda$ has a Galois-saturated extension $M' \in \mathcal{K}_\lambda$. Then $\mathcal{K}$ is λ-Galois stable.*

Remarkably, the existence of Galois-saturated extensions of the sort required in the proposition above can be guaranteed by the purely category-theoretic condition known as weak λ-stability. It is a still more remarkable fact that weak stability occurs in many cardinalities in any accessible category hence also, as we will see, in any AEC. This leads, in Section 7, to a partial stability spectrum result for weakly tame AECs.

3. Accessible categories

Of the basic properties that we retain in passing to abstract elementary classes from classes of structures born of syntactic considerations (classes of models of first order theories, sentences in $L_{\kappa,\omega}$, $L_{\omega_1,\omega}(Q)$, and so on), two stand out as being of particular importance. First, the union axioms ensure that the class is closed under unions of chains, giving us the structure needed to run certain nearly-classical model-theoretic arguments. Moreover, the Downward Löwenheim–Skolem Property for AECs guarantees that any structure $M \in \mathcal{K}$ can be obtained as the directed union of its submodels of cardinality at most $\mathrm{LS}(\mathcal{K})$, meaning that an AEC $\mathcal{K}$ is, in fact, generated from the set of all such small models, $\mathcal{K}_{\mathrm{LS}(\mathcal{K})}$. Although accessible categories – the category theorists' preferred generalization of classes of structures (see [1,12]) – involve a slightly greater degree of abstraction and hence greater generality, they are also characterized by precisely these two traits: each accessible category is closed under certain highly directed colimits (if not arbitrary directed colimits), and is generated from a set of "small" objects.

To flesh out what we mean by "small," we require a notion of size that makes sense in an arbitrary category. Since, in particular, we do not wish to restrict ourselves to categories of structured sets, our notion will need to be more subtle than mere cardinality. The solution to this quandary – presentability – is treated admirably in [1] and [12]. We begin with the simplest and most mathematically natural case:

Definition 3.1. An object N in a category $\mathbf{C}$ is said to be *finitely presentable* (or ω-*presentable*) if the corresponding hom-functor $\mathrm{Hom}_{\mathbf{C}}(N, -)$ preserves directed colimits.

Less cryptically, N is finitely presentable if for any directed poset I and diagram $D : (I, \leq) \to \mathbf{C}$ with colimit cocone $(\phi_i : D(i) \to M)_{i \in I}$, any map $f : N \to M$ factors through one of the maps in the colimit cocone: $f = \phi_i \circ g$ for some $i \in I$ and $g : N \to D(i)$, as in the diagram below.

$$\begin{tikzcd}
 & & M & \\
 & D(i) \arrow[ur, "\phi_i"] \arrow[rr, "D(i \to j)"'] & & D(j) \arrow[ul, "\phi_j"'] \\
N \arrow[ur, dashed, "g"'] \arrow[uurr, bend left, "f"] & & &
\end{tikzcd}$$

Moreover, this factorization must be essentially unique, in the sense that for any two such, say g and g' from N to $D(i)$ with $f = \phi_i \circ g = \phi_i \circ g'$, there is a $j \geq i$ in I such that $D(i \to j) \circ g = D(i \to j) \circ g'$.

Examples:

1. In **Set**, the category of sets, an object X is finitely presentable if and only if it is a finite set.
2. Let Σ be a finitary relational signature, and $\mathbf{Rel}(\Sigma)$ the category of Σ-structures and maps that preserve the relations $R \in \Sigma$. An object M in $\mathbf{Rel}(\Sigma)$ is finitely presentable if and only if $|M|$ is finite and there are only finitely many Σ-edges in M: $\sum_{R\in\Sigma} |R^M| < \aleph_0$.
3. In **Grp**, the category of groups and group homomorphisms, an object G is finitely presentable if and only if it is finitely presented in the usual sense: G has finitely many generators subject to finitely many relations.
   As shown in [1], the same holds in any variety of finitary algebras.

Many more examples can be found in [1]. One more word about the category **Grp**: every object of **Grp** – every group – can be obtained as the directed colimit of finitely presented groups, hence as a directed colimit of finitely presentable objects. Moreover, **Grp** is closed under arbitrary directed colimits. This means, in short, that **Grp** is a finitely accessible category. The precise definition:

Definition 3.2. A category **C** is *finitely accessible* (or *ω-accessible*) if

- **C** contains only a set of finitely presentable objects up to isomorphism, and every object in **C** is a directed colimit of finitely presentable objects.
- **C** is closed under directed colimits.

Finitely accessible categories abound in mainstream mathematics: the category **Grp** (or, indeed, any category of finitary algebraic varieties), $\mathbf{Rel}(\Sigma)$, **Set**, and so on.

The notions of finite presentability and finite accessibility generalize in a natural fashion. Let λ be an infinite regular cardinal. We first recall:

Definition 3.3. 1. A poset I is said to be *λ-directed* if for every subset $X \subseteq I$ of cardinality less than λ, there is an element $i \in I$ such that for every $x \in X$, $x \leq i$.
2. A colimit in a category **C** is λ-directed if it is the colimit of a λ-directed diagram; that is, a diagram of the form $D : (I, \leq) \to \mathbf{C}$, where I is a λ-directed poset.

Generalizing finitely presentable objects, we define:

Definition 3.4. An object N is said to be *λ-presentable* if the corresponding functor $\mathrm{Hom}(N, -)$ preserves λ-directed colimits.

We may unravel this definition as before: N is λ-presentable if for any λ-directed poset I and diagram $D : (I, \leq) \to \mathbf{C}$ with colimit cocone $(\phi_i : D(i) \to M)_{i\in I}$, any map $f : N \to M$ factors essentially uniquely through one of the maps in the colimit cocone: $f = \phi_i \circ g$ for some $i \in I$ and some $g : N \to D(i)$ (as in the diagram following Definition 3.1 above).

For any category **C** and infinite regular cardinal λ, we denote by $\mathbf{Pres}_\lambda(\mathbf{C})$ a full subcategory of **C** consisting of one representative of each isomorphism class of λ-presentable objects; that is, $\mathbf{Pres}_\lambda(\mathbf{C})$ is a skeleton of the full subcategory consisting of all λ-presentable objects.

One should note that it is customary – and sometimes advantageous – to phrase things in terms of λ-filtered (rather than λ-directed) diagrams and colimits, but the two characterizations are fundamentally equivalent. See, in particular, Remark 1.21 in [1]. Now, the crucial definition:

Definition 3.5. 1. Let λ be an infinite regular cardinal. A category **C** is *λ-accessible* if
   - **C** contains only a set of λ-presentable objects up to isomorphism, and every object in **C** is a λ-directed colimit of λ-presentable objects.
   - **C** is closed under λ-directed colimits
2. We say that a category **C** is *accessible* if it is λ-accessible for some λ.

A natural question: If a category is λ-accessible, will it be accessible in regular cardinals $\mu \geq \lambda$ and, if so, in which of these cardinals? As it happens, there is a necessary and sufficient condition for upward transfer of accessibility: a λ-accessible category is accessible in $\mu > \lambda$ precisely when $\mu \trianglerighteq \lambda$, where $\trianglerighteq$ denotes the sharp inequality relation. We leave the characterization of $\trianglerighteq$ to Theorem 2.11 in [1]. The critical point is that for each set of regular cardinals L, there are arbitrarily large regular cardinals μ with the property that $\lambda \triangleleft \mu$ for all $\lambda \in L$. This will play an important role in the partial stability spectrum result in Section 7.

4. AECs as accessible categories

In this section and the two that follow, we make no global assumptions regarding amalgamation or joint embedding. We will explicitly indicate the few scattered results that do in fact require these properties. Given an AEC $\mathcal{K}$, we regard it as a category in the natural way: the objects are the models $M \in \mathcal{K}$, and the morphisms are precisely the strong embeddings. Since there is no serious risk of confusion, we will also refer to the category thus obtained as $\mathcal{K}$. The first step in our analysis of the connections between AECs and accessible categories is the following:

Theorem 4.1. *Let $\mathcal{K}$ be an AEC. Then $\mathcal{K}$ is μ-accessible for every regular cardinal $\mu > LS(\mathcal{K})$. In particular, $\mathcal{K}$ is $LS(\mathcal{K})^+$-accessible.*

Our task, then, is to show that for each regular cardinal $\mu > \mathrm{LS}(\mathcal{K})$, $\mathcal{K}$ contains a set (up to isomorphism) of μ-presentable objects, every model in $\mathcal{K}$ can be obtained as a μ-directed colimit of μ-presentable objects, and $\mathcal{K}$ is closed under μ-directed colimits. We accomplish this through a series of easy lemmas. First:

Lemma 4.2. *Let $M \in \mathcal{K}$. For any regular $\mu > LS(\mathcal{K})$, M is a μ-directed union of its strong submodels of size less than μ.*

Proof. Consider the diagram consisting of all submodels of M of size less than μ and with arrows the strong inclusions. To check that this diagram is μ-directed, we must show that any collection of fewer than μ many such submodels have a common extension also belonging to the diagram. Let $\{M_\alpha \mid \alpha < \nu\}$, $\nu < \mu$, be such a collection. Since μ is regular, $\sup\{|M_\alpha| \mid \alpha < \nu\} < \mu$, whence

$$\left|\bigcup_{\alpha<\nu} M_\alpha\right| \leq \nu \cdot \sup\{|M_\alpha| \mid \alpha < \nu\} < \nu \cdot \mu = \mu$$

This set will be contained in some $M' \prec_{\mathcal{K}} M$ with $|M'| < \mu$, by the Downward Löwenheim–Skolem Property. For each $\alpha < \nu$, $M_\alpha \prec_{\mathcal{K}} M$ and $M_\alpha \subseteq M'$. Since $M' \prec_{\mathcal{K}} M$, coherence implies that $M_\alpha \prec_{\mathcal{K}} M'$. So we are done. □

Lemma 4.3. *For any regular cardinal $\mu > LS(\mathcal{K})$, a model $M \in \mathcal{K}$ is μ-presentable if and only if $|M| < \mu$. In particular, M is $LS(\mathcal{K})^+$-presentable if and only if $|M| \leq LS(\mathcal{K})$.*

Proof. ($\Rightarrow$) Suppose that M is μ-presentable, and consider the identity map $M \hookrightarrow_{\mathcal{K}} M$. As we saw in the previous lemma, M is a μ-directed union of its submodels of size strictly less than μ. By μ-presentability of M, the identity map factors through one of the inclusions $M' \hookrightarrow_{\mathcal{K}} M$ in the colimit cocone. Since all maps in the category are injective, M can have cardinality no greater than that of the model M'. Hence $|M| < \mu$.

($\Leftarrow$) Suppose $|M| = \nu < \mu$. Let M' be a μ-directed colimit, say

$$M' = \mathrm{Colim}_{i \in I} M_i$$

with I a μ-directed poset, connecting maps $\phi_{ij} : M_i \hookrightarrow_{\mathcal{K}} M_j$ for $i \leq j$, and colimit cocone maps $\phi_i : M_i \hookrightarrow_{\mathcal{K}} M'$. That is, for each $i \leq j$ in I, we have the commutative triangle

$$\begin{tikzcd} & M' & \\ M_i \arrow[ur, "\phi_i"] \arrow[rr, "\phi_{ij}"'] & & M_j \arrow[ul, "\phi_j"'] \end{tikzcd}$$

Consider an embedding $f : M \hookrightarrow_{\mathcal{K}} M'$. The image $f[M]$ is a strong submodel of M, and is of cardinality $\nu < \mu$. Since $\mathcal{K}$ is a concrete category, the submodels $\phi_i[M_i]$ of M' cover M', meaning that for each $m \in f[M]$ we may choose a $\phi_{i_m}[M_{i_m}]$ containing it. By μ-directedness of I, there is a $j \in I$ with $j \geq i_m$ for all $m \in f[M]$. By the commutativity condition above, one can see that $\phi_{i_m}[M_{i_m}] \subseteq \phi_j[M_j]$ for all m, meaning that $f[M] \subseteq \phi_j[M_j]$ and, by coherence, $f[M] \prec_{\mathcal{K}} \phi_j[M_j]$. Hence the embedding $f : M \hookrightarrow_{\mathcal{K}} M'$ factors through $\phi_j : M_j \hookrightarrow_{\mathcal{K}} M'$ as

$$M \xrightarrow{\phi_j^{-1} \circ f} M_j \xrightarrow{\phi_j} M'.$$

This factorization is unique: for any other factorization map $g : M \to M_j$, we have $\phi_j \circ (\phi_j^{-1} \circ f) = \phi_j \circ g$ and, since ϕ_j is a monomorphism, it follows that $\phi_j^{-1} \circ f = g$. This means, of course, that M is μ-presentable. □

The punchline of all this is:

Lemma 4.4. *For any regular $\mu > LS(\mathcal{K})$, $\mathcal{K}$ contains a set of μ-presentables, namely $\mathcal{K}_{<\mu}$, and every model in $\mathcal{K}$ is a μ-directed colimit of objects in $\mathcal{K}_{<\mu}$.*

By an easy exercise in universal algebra, Axiom (A2) – closure under unions of chains – implies closure under arbitrary directed colimits. Since every μ-directed diagram is, in particular, directed, we can complete the proof of the theorem:

Lemma 4.5. *For any regular cardinal $\mu > LS(\mathcal{K})$, $\mathcal{K}$ is closed under μ-directed colimits.*

One often encounters (here and elsewhere) the claim that AECs are the result of extracting the purely category-theoretic content of elementary classes, preserving the essence of the elementary submodel relation while dispensing with syntax and certain properties – such as compactness – that are typically derived from the ambient logic. We obtain very definite confirmation of this claim if we compare Theorem 4.1 above with the following result of [13]:

Proposition 4.6. *Given a first order theory T in language $L(T)$, let* $\mathbf{Elem}(T)$ *be the category with objects the models of T and morphisms the elementary embeddings. Then for any regular $\mu > |L(T)|$,* $\mathbf{Elem}(T)$ *is μ-accessible, and $M \in \mathcal{K}$ is μ-presentable if and only if $|M| < \mu$.*

We now consider the way in which an AEC $\mathcal{K}$ in signature L sits inside what is, for model theorists, the natural ambient category of L-structures: L-**Struct**, whose objects are L-structures and whose morphisms are injective L-homomorphisms that both preserve and reflect the relations in L. The goal is to produce a category-theoretic axiomatization that, in any such category L-**Struct**, picks out all the subcategories corresponding to AECs in the signature L. It is, of course, possible to pick out AEC-like subcategories in more general frameworks, as in the "base categories" of [9] or, in [4], in an arbitrary finitely accessible category. Naturally, all three axiomatizations are fundamentally equivalent. Our emphasis on the concrete tends to align us more closely with the former, differing insofar as we condense a number of axioms from [9] under the heading of accessibility, thereby making clear the connection between AECs and the existing body of work on accessible categories. This perspective clarifies, for example, that the abstract notion of size laid out in [9] corresponds to the well-established notion of presentability.

We introduce two definitions:

Definition 4.7. Fix a category $\mathbf{B}$ and subcategory $\mathbf{C}$.

- We say that $\mathbf{C}$ is a *replete* subcategory of $\mathbf{B}$ if for every M in $\mathbf{C}$ and every isomorphism $f : M \to N$ in the larger category $\mathbf{B}$, both f and N are in $\mathbf{C}$.
- We say that $\mathbf{C}$ is a *nearly full* subcategory of $\mathbf{B}$ if for every commutative diagram

$$\begin{tikzcd}
& M_2 \arrow[dr, "h"] & \\
M_1 \arrow[ur, dashed, "f"] \arrow[rr, "g"'] & & N
\end{tikzcd}$$

with h and g (hence also their domains and codomains) in $\mathbf{C}$ and with f in $\mathbf{B}$, then in fact f is in $\mathbf{C}$.

Notice that the second property corresponds exactly to the coherence axiom for AECs. Indeed, subcategories of this form are referred to as "coherent" in [9,10]—the term "nearly full" was introduced in [4] as an alternative to this already overburdened word.

Purely from Theorem 4.1 and the axioms for AECs,

Proposition 4.8. *An AEC $\mathcal{K}$ is a replete, nearly full subcategory of $L(\mathcal{K})$-*$\mathbf{Struct}$ *which is μ-accessible for all $\mu > LS(\mathcal{K})$ and has all directed colimits. Moreover, the directed colimits are computed as in $L(\mathcal{K})$-*$\mathbf{Struct}$*.*

Now, consider a category L-**Struct**, L a finitary signature, consisting of L-structures and injective L-homomorphisms, as before. The natural question: given a replete, coherent subcategory of L-**Struct** with all directed colimits (computed as in L-**Struct**) that is μ-accessible for all μ strictly larger than some cardinal λ, can it be regarded as an AEC? The answer is yes: for any such subcategory $\mathbf{C}$, consider the class consisting of its objects (call it $\mathbf{C}$ as well), with relation $\prec_{\mathbf{C}}$ defined by the condition that $M \prec_{\mathbf{C}} N$ if and only if $M \subseteq_L N$ and the inclusion map is a $\mathbf{C}$-morphism.

Theorem 4.9. *The class* $\mathbf{C}$ *is an AEC.*

Proof. The relation $\prec_{\mathbf{C}}$ is transitive, and certainly refines the substructure relation. Coherence and closure under isomorphism hold by assumption, and the union of chains axioms are easily verified as well. As for the Löwenheim–Skolem property, let $M \in \mathbf{C}$, and let $A \subseteq M$. Consider $\mu = |A| + \lambda$. The cardinal μ^+ is regular and $\mu^+ > \lambda$, meaning that $\mathbf{C}$ is μ^+-accessible. This means, in turn, that every object M is a μ^+-filtered colimit of μ^+-presentable objects, and thus the μ^+-directed union of the images of these μ^+-presentable objects under the cocone maps. All of these images are, of course, strong submodels of M. Since $|A| \leq |A| + \lambda < \mu^+$, the μ^+-directedness of the union implies that A is contained in one of the structures in the union, say N. As N is μ^+-presentable, it is, by the proof of the "only if" direction of Lemma 4.3 above, of cardinality at most $\mu = |A| + \lambda$. One can see, then, that λ will do as LS($\mathbf{C}$). $\square$

The amalgamation and joint embedding properties for AECs are purely diagrammatic, and coincide exactly with the analogues for accessible categories included in [13]. If we add them to the axioms in Proposition 4.8, we obtain an axiomatization of AECs with the AP and JEP. On the other hand, if we replace L-**Struct** with a particular category of metric L-structures (as in [9]), our axioms describe the abstract metric classes in the signature L.

5. Model theory and category theory: correspondences

We turn now to the task of providing a dictionary between the language of accessible categories and that of AECs. We will primarily be interested in examining the translations of category-theoretic notions originally defined in [13]. The latter piece is, of course, concerned with accessible categories with directed colimits—which are almost AECs, as we now know.

We begin with the easiest correspondence. Accompanying our notion of size for objects in accessible categories – presentability – is a natural notion of categoricity:

Definition 5.1. A category **C** is λ-*categorical* if it contains, up to isomorphism, a unique object N which is λ^+-presentable, but not μ-presentable for any $\mu < \lambda^+$. **C** is said to be *strongly* λ-*categorical* if it contains, up to isomorphism, a unique λ^+-presentable object.

From Lemma 4.3, we have:

Proposition 5.2. *For an AEC $\mathcal{K}$, λ-categoricity of the corresponding category is equivalent to λ-categoricity in the usual sense. $\mathcal{K}$ is strongly λ-categorical if and only if it contains only a single model of size less than λ^+ (up to isomorphism).*

Before we proceed, we lay out two basic facts that will come in handy in simplifying the diagrams that crop up in our investigations, allowing us to replace certain strong embeddings by strong inclusions.

Remark 5.3. 1. Any strong embedding $f : M_0 \hookrightarrow_{\mathcal{K}} M$ factors as an isomorphism $M_0 \hookrightarrow_{\mathcal{K}} f[M_0]$ followed by the strong inclusion $f[M_0] \prec_{\mathcal{K}} M$.
2. Given a strong embedding $f : M_0 \hookrightarrow_{\mathcal{K}} M$, there is an extension M_1 of M_0 isomorphic to M. Moreover, we may take the isomorphism $g : M \hookrightarrow_{\mathcal{K}} M_1$ to be inverse to f on $f[M_0]$; that is, $g \circ f : M_0 \hookrightarrow_{\mathcal{K}} M_1$ fixes M_0.

Now we may begin. Unless otherwise specified, λ is understood to be a regular cardinal. We first consider λ-saturation of the sort introduced in [13]:

Definition 5.4. An object M in a category **C** is said to be λ-*saturated* if for any λ-presentable objects N, N' and morphisms $f : N \to M$ and $g : N \to N'$, there is a morphism $h : N' \to M$ such that the following diagram commutes:

$$\begin{tikzcd} & M & \\ N \arrow[ur, "f"] \arrow[rr, "g"'] & & N' \arrow[ul, dashed, "h"'] \end{tikzcd}$$

This looks more like a homogeneity condition, although it matches up nicely with the classical notion of λ-saturation in elementary classes. For AECs, we have:

Proposition 5.5. *For any AEC $\mathcal{K}$ and $M \in \mathcal{K}$, M is λ-saturated if and only if it is λ-model homogeneous.*

Proof. ($\Rightarrow$) Let $N \prec_{\mathcal{K}} M$ and $N \prec_{\mathcal{K}} N'$, with $|N|$ and $|N'|$ strictly less than λ. Notice that, by Lemma 4.3, N and N' are λ-presentable. Then, by λ-saturation of M, there is a strong embedding $h : N' \hookrightarrow_{\mathcal{K}} M$ such that the following diagram commutes:

$$\begin{tikzcd} & M & \\ N \arrow[ur] \arrow[rr] & & N' \arrow[ul, dashed, "h"'] \end{tikzcd}$$

with $N \hookrightarrow_{\mathcal{K}} N'$ and $N \hookrightarrow_{\mathcal{K}} M$ the inclusions. This says precisely that h is an embedding of N' into M fixing N. So M is λ-model homogeneous.

($\Leftarrow$) Using one application of each of the facts in Remark 5.3, one can see that it suffices to consider diagrams of strong inclusions

$$\begin{tikzcd} & M & \\ N \arrow[ur] \arrow[rr] & & N' \end{tikzcd}$$

with N and N' both λ-presentable. Then $N, N' \in \mathcal{K}_{<\lambda}$ and λ-model homogeneity of M gives a strong embedding $h : N' \hookrightarrow_{\mathcal{K}} M$ that fixes N, making the diagram commute. $\square$

Recalling that λ-model homogeneity is equivalent to λ-Galois-saturation in AECs with amalgamation and joint embedding (see Proposition 2.7), we get

Corollary 5.6. *For any AEC $\mathcal{K}$ with the AP and JEP and any $\lambda > LS(\mathcal{K})$, a model $M \in \mathcal{K}$ is λ-saturated if and only if M is λ-Galois-saturated.*

Definition 5.7. Let λ be a regular cardinal. A morphism $f : M \to M'$ in a category $\mathbf{C}$ is said to be λ-*pure* if for any commutative square

$$\begin{tikzcd} N \arrow[r, "g"] \arrow[d, "u"'] & N' \arrow[d, "v"] \arrow[dl, dashed, "h"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

in which N and N' are λ-presentable, there is a morphism $h : N' \to M$ such that $h \circ g = u$.

In elementary classes, one can show that an elementary inclusion of a model M in a model M' is λ-pure only if for every $A \subseteq M$ with $|A| < \lambda$ and every $p \in S(A)$, if p is realized in M', then it is also realized in M. That is, M is λ-saturated relative to M'. We will obtain a similar result for AECs, but first note that relative λ-model homogeneity is a more obvious analogue:

Proposition 5.8. *For $\lambda \geq LS(\mathcal{K})$, a strong inclusion $M \prec_{\mathcal{K}} M'$ is λ-pure if and only if M is λ-model homogeneous relative to M': for any $N \prec_{\mathcal{K}} M$ and $N \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} M'$ with $N, N' \in \mathcal{K}_{<\lambda}$, there is an embedding of N' into M fixing N.*

Proof. ($\Rightarrow$) Suppose that $M \hookrightarrow_{\mathcal{K}} M'$ is a λ-pure inclusion. Let $N \prec_{\mathcal{K}} M$ with $|N| < \lambda$ and N' with $N \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} M'$ and $|N'| < \lambda$. The various inclusions yield a commutative square of the form in the definition above. By λ-purity of the bottom inclusion, there is a strong embedding $h : N' \hookrightarrow_{\mathcal{K}} M$ that makes the upper triangle of

$$\begin{tikzcd} N \arrow[r] \arrow[d] & N' \arrow[d] \arrow[dl, "h"] \\ M \arrow[r] & M' \end{tikzcd}$$

commute. This commutativity condition means that $h : N' \hookrightarrow_{\mathcal{K}} M$ fixes N, so we are done.

($\Leftarrow$) By Remark 5.3 and a little diagram-wrangling, it suffices to consider commutative squares of strong inclusions (as in the diagram above). With this reduction, the proof becomes trivial: for any $N \prec_{\mathcal{K}} M$ with $|N| < \lambda$ and N' with $N \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} M'$ and $|N'| < \lambda$, if M is λ-model homogeneous relative to M', there is an embedding $h : N' \to M$ that fixes N. As above, this guarantees that h satisfies the appropriate commutativity condition, hence witnessing λ-purity of the inclusion $M \hookrightarrow_{\mathcal{K}} M'$. $\square$

Proposition 5.9 (*AP, JEP*)**.** *For $\lambda > LS(\mathcal{K})$, a strong inclusion $M \prec_{\mathcal{K}} M'$ is λ-pure only if M is λ-Galois-saturated relative to M': every type over $N \prec_{\mathcal{K}} M$ with $|N| < \lambda$ that is realized in M' is realized in M.*

Proof. By Proposition 5.8, λ-purity of $M \hookrightarrow_{\mathcal{K}} M'$ implies that M is λ-model homogeneous relative to M'. Let $N \prec_{\mathcal{K}} M$, $N \in \mathcal{K}_{<\lambda}$, and let p be any type over N that is realized in M', say by a. Take $N' \prec_{\mathcal{K}} M'$ containing $N \cup \{a\}$, $N' \in \mathcal{K}_{<\lambda}$. By relative λ-model homogeneity, there is an embedding $h : N' \hookrightarrow_{\mathcal{K}} M$ that fixes N. Thus any extension of h to an automorphism of $\mathfrak{C}$ (the existence of which follows from strong model homogeneity of $\mathfrak{C}$) lies in $\mathrm{Aut}_N(\mathfrak{C})$, and witnesses that a and $h(a)$ have the same Galois type over N. Since $h(a) \in M$, we are done. $\square$

As a first step in generalizing from λ-purity of strong inclusions to arbitrary strong embeddings, the fact that compositions of isomorphisms with λ-pure maps are λ-pure implies:

Proposition 5.10. *A strong embedding $f : M \hookrightarrow_{\mathcal{K}} M'$ is λ-pure if and only if the inclusion $f[M] \prec_{\mathcal{K}} M'$ is λ-pure.*

We may now characterize λ-purity for arbitrary strong embeddings.

Corollary 5.11. *For $\lambda > LS(\mathcal{K})$, a strong embedding $f : M \hookrightarrow_{\mathcal{K}} M'$ is λ-pure if and only if $f[M]$ is λ-model homogeneous relative to M'. Assuming AP and JEP, a strong embedding $f : M \hookrightarrow_{\mathcal{K}} M'$ is λ-pure only if $f[M]$ is λ-Galois-saturated relative to M'.*

Moreover, since the monster model $\mathfrak{C}$ is both Galois-saturated and model homogeneous, we have:

Proposition 5.12 (*AP, JEP*)**.** *For $\lambda > LS(\mathcal{K})$, a strong embedding $M \hookrightarrow_{\mathcal{K}} \mathfrak{C}$ is λ-pure if and only if M is λ-model homogeneous (or, equivalently, λ-Galois-saturated).*

We turn now to the property most indispensable for our purposes: weak λ-stability.

Definition 5.13. Let λ be a regular cardinal. A category $\mathbf{C}$ is said to be *weakly λ-stable* if for any λ^+-presentable M and morphism $f : M \to M'$, f factors as $M \xrightarrow{g} \bar{M} \xrightarrow{h} M'$, where $\bar{M}$ is λ^+-presentable, and h is λ-pure.

In the elementary case, weak λ-stability of $\mathbf{Elem}(T)$ follows from λ-stability of the first order theory T (see the discussion following Definition 2.31 in [10]). Although it is not clear whether λ-Galois stability of an AEC implies weak λ-stability of the associated category, it is still possible to give a reasonably model-theoretic condition sufficient to guarantee the latter property:

Theorem 5.14. *If λ is a regular cardinal, $\lambda > LS(\mathcal{K})$, and for all $N \in \mathcal{K}_{<\lambda}$, N has fewer than λ strong extensions of size less than λ (up to isomorphism over N), $\mathcal{K}$ is weakly λ-stable.*

Proof. Let $M \in \mathcal{K}_{\leq\lambda}$ and $M' \in \mathcal{K}$ with $M \prec_{\mathcal{K}} M'$. We construct an intermediate extension $\bar{M} \in \mathcal{K}_{\leq\lambda}$ witnessing weak λ-stability of $\mathcal{K}$. To begin, enumerate M as $\{a_i \mid i < |M|\}$. We construct $\bar{M}$ as the union of a continuous $\prec_{\mathcal{K}}$-increasing chain of length λ consisting of models of size less than λ, each of which collects small extensions of the preceding ones. In detail, we proceed as follows:

- $i = 0$: Take $N_0 \prec_{\mathcal{K}} M$, $|N_0| < \lambda$.
- $i = j + 1$: We have $N_j \prec_{\mathcal{K}} M'$, $|N_j| < \lambda$. By our assumption, N_j has fewer than λ many strong extensions in M' of size strictly less than λ, up to isomorphism over N_j. Select a representative from each isomorphism class. The union of N_j and all such representatives is of size strictly less than λ (by regularity). Hence we may take a model $N_i \prec_{\mathcal{K}} M'$ with $|N_i| < \lambda$ that contains the aforementioned union and the element a_j. Notice that $N_j \prec_{\mathcal{K}} N_i$, by coherence.
- i limit: Define $N_i = \bigcup_{j<i} N_j$. Notice that $|N_\alpha| < \lambda$, by regularity of λ.

Let $\bar{M} = \bigcup_{i<\lambda} N_i$. Notice that $\bar{M} \subseteq M'$, whence $\bar{M} \prec_{\mathcal{K}} M'$ by coherence.

In order to see that $\bar{M}$ has the desired property, let $N \prec_{\mathcal{K}} M$, $|N| < \lambda$, and let N' be a model of size less than λ with $N \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} M'$. By regularity of λ, $N \subseteq N_i$ for some $i < \lambda$. Take a model $N'' \prec_{\mathcal{K}} M'$ of size less than λ that contains $N_i \cup N'$. This is an extension of N_i in M' of cardinality less than λ, and so, by construction, there is an N''' collected in the next model in the chain, N_{i+1}, that is isomorphic to N'' over N_i, say via $f : N'' \hookrightarrow_{\mathcal{K}} N'''$. In particular, the isomorphism f fixes $N \prec_{\mathcal{K}} N_i$. Now, $N''' \subseteq N_{i+1} \prec_{\mathcal{K}} \bar{M}$ and, by coherence, $N''' \prec_{\mathcal{K}} \bar{M}$. The composition

$$N' \prec_{\mathcal{K}} N'' \overset{f}{\hookrightarrow}_{\mathcal{K}} N''' \prec_{\mathcal{K}} \bar{M}$$

is the desired embedding of N' in $\bar{M}$ fixing N. $\square$

More interesting for our purposes is the converse, roughly speaking: the way in which weak λ-stability controls the proliferation of Galois types over models in AECs.

Proposition 5.15. *(AP, JEP) For any $\lambda \geq LS(\mathcal{K})$, if $\mathcal{K}$ is weakly λ-stable, then any $M \in \mathcal{K}_\lambda$ has a Galois-saturated extension $\bar{M} \in \mathcal{K}_\lambda$.*

Proof. If $M \in \mathcal{K}_\lambda$, it is λ^+-presentable. Hence the embedding $M \hookrightarrow_{\mathcal{K}} \mathfrak{C}$ factors as

$$M \hookrightarrow_{\mathcal{K}} \bar{M} \hookrightarrow_{\mathcal{K}} \mathfrak{C}$$

where $\bar{M}$ is λ^+-presentable (meaning that $\bar{M} \in \mathcal{K}_{\leq\lambda}$, and since $|\bar{M}| \geq |M| = \lambda$, $\bar{M} \in \mathcal{K}_\lambda$) and the second map in the factorization is λ-pure. By Proposition 5.12, the claim follows. $\square$

The condition in the consequent of Proposition 5.15 – the existence of Galois-saturated extensions of size λ – is precisely the condition from which we are able, by Theorem 2.11, to conclude λ-stability in a weakly χ-tame and μ-totally transcendental AEC, provided $\lambda \geq \chi$ and $\mathrm{cf}(\lambda) > \mu$. That is, weak λ-stability actually implies full λ-Galois stability in this context. This fact lies at the heart of the spectrum result in Section 7.

6. A structure theorem for categorical AECs

In [13], Jiří Rosický proves a structure theorem for strongly λ-categorical λ^+-accessible categories, which has, as an interesting consequence, a structure theorem for large models in categorical AECs. The result and the argument we give, it should be stressed, are special cases of those in [13], yet the former is completely novel, model-theoretically speaking. We present the argument in considerable detail, primarily as an illustration of what it means, in practical terms, to work on AECs using the category-theoretic toolkit, and as evidence that the category-theoretic perspective can be genuinely useful in illuminating their essential structure.

Let $\mathcal{K}$ be a λ-categorical AEC. Denote by $\mathcal{K}'$ the class $\mathcal{K}_{\geq\lambda}$, with $\prec_{\mathcal{K}'}$ simply the restriction of $\prec_{\mathcal{K}}$. Notice that $\mathcal{K}'$ is still an AEC, albeit with $\mathrm{LS}(\mathcal{K}') = \lambda$. Notice also that $(\mathcal{K}', \prec_{\mathcal{K}'})$ is precisely the same category as $(\mathcal{K}_{\geq\lambda}, \prec_{\mathcal{K}})$: same objects, same morphisms. It is λ^+-accessible (by Theorem 4.1), and strongly λ-categorical in the sense of Definition 5.1. Let C be a representative of the unique isomorphism class in cardinality λ, and note that $(\mathcal{K}')_\lambda$ is equivalent to the one object category consisting of C and the set of its endomorphisms. We use M to refer both to this one object category and to the corresponding monoid, where the multiplication is given by postcomposition: $f \cdot g = f \circ g$. We will show that $\mathcal{K}'$ is equivalent to a highly structured subcategory of the category of sets with M-actions.

First, we fix our terminology:

Definition 6.1. Let M be a monoid. An *M-set* is a pair (X, ρ), where X is a set and $\rho : M \times X \to X$ is an action (which we typically write using product notation) satisfying the following conditions for all $a, b \in M$ and $x \in X$: $1 \cdot x = x$ and $(ab) \cdot x = a \cdot (b \cdot x)$.

A map $h : (X_1, \rho_1) \to (X_2, \rho_2)$ is an *M-set homomorphism* if for all $a \in M$ and $x \in X$, $h(a \cdot x) = a \cdot h(x)$, where the actions on the left and right hand sides of the equation are ρ_1 and ρ_2, respectively.

Definition 6.2. Let M be a monoid. We denote by M**-Set** the category of M-sets and M-set homomorphisms. For any regular cardinal λ, we denote by (M, λ)**-Set** the full subcategory of M**-Set** consisting of all λ-directed colimits of copies of M, where the latter is considered as an M-set in the obvious way.

Recall also the notion of equivalence with which we will be working:

Definition 6.3. An *equivalence* between categories **C** and **D** is given by a pair of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ with natural isomorphisms $F \circ G \simeq 1_{\mathbf{D}}$ and $G \circ F \simeq 1_{\mathbf{C}}$. Under these conditions, the functors F and G are referred to as *equivalences of categories*.

Any equivalence of categories $F : \mathbf{C} \to \mathbf{D}$ is full and faithful (bijective on Hom-sets), and essentially surjective: for any object D in **D**, there is an object C in **C** with $F(C) \simeq D$. In short, equivalent categories are structurally identical, as long as we are interested in objects only up to isomorphism.

We now produce the desired equivalence. Recall that for any category **C**, the category of presheaves on **C**, denoted $\mathbf{Set}^{\mathbf{C}^{op}}$, consists of all contravariant **Set**-valued functors on **C** and all natural transformations between them. First, we show:

Lemma 6.4. *The AEC $\mathcal{K}'$ is equivalent to the full subcategory of $\mathbf{Set}^{M^{op}}$ consisting of λ^+-directed colimits of $Hom_{\mathcal{K}'}(-, C)$.*

Proof. We define a functor $F : \mathcal{K}' \to \mathbf{Set}^{M^{op}}$ as follows: for any N in $\mathcal{K}'$,

$$F(N) = \mathrm{Hom}_{\mathcal{K}'}(-, N)$$

is the functor that takes C to $F(N)(C) = \mathrm{Hom}_{\mathcal{K}'}(C, N)$ and takes any endomorphism $g : C \hookrightarrow_{\mathcal{K}} C$ to the set map $F(N)(g) : \mathrm{Hom}_{\mathcal{K}'}(C, N) \to \mathrm{Hom}_{\mathcal{K}'}(C, N)$ that sends each $h \in \mathrm{Hom}_{\mathcal{K}'}(C, N)$ to $h \circ g$. The equivalence F takes any strong embedding $f : N \to N'$ to the map $F(f) : \mathrm{Hom}_{\mathcal{K}'}(-, N) \to \mathrm{Hom}_{\mathcal{K}'}(-, N')$, where $F(f)(g) = f \circ g$ for any $g \in \mathrm{Hom}_{\mathcal{K}'}(C, N)$. We must show that every object in the image of F is (isomorphic to) a λ^+-directed colimit of copies of $\mathrm{Hom}_{\mathcal{K}'}(-, C)$. To begin, any $N \in \mathcal{K}'$ is a λ^+-directed colimit of copies of C, say $N = \mathrm{Colim}_{i \in I} C$. By λ^+-presentability of C,

$$\mathrm{Hom}_{\mathcal{K}'}(C, N) = \mathrm{Hom}_{\mathcal{K}'}(C, \mathrm{Colim}_{i \in I} C) \simeq \mathrm{Colim}_{i \in I} \mathrm{Hom}_{\mathcal{K}'}(C, C)$$

meaning that

$$\mathrm{Hom}_{\mathcal{K}'}(-, N) \simeq \mathrm{Colim}_{i \in I} \mathrm{Hom}_{\mathcal{K}'}(-, C)$$

as functors on the category M, which has C as its only object.

Similar considerations yield the functor G in the other direction, which forms the second part of the equivalence. Any H in the subcategory of $\mathbf{Set}^{M^{op}}$ in which we are interested is a λ^+-directed colimit of copies of $\mathrm{Hom}_{\mathcal{K}'}(-, C)$, say

$$H = \mathrm{Colim}_{i \in I} \mathrm{Hom}_{\mathcal{K}'}(-, C)$$

where the maps in the I-indexed diagram are natural transformations $\phi_{ij} : \mathrm{Hom}_{\mathcal{K}'}(-, C) \to \mathrm{Hom}_{\mathcal{K}'}(-, C)$ for $i \leq j$ in I. By the Yoneda Lemma, the functor F is full and faithful, meaning that this diagram arises (morphisms and all) from an I-indexed diagram in $\mathcal{K}'$.

By λ^+-presentability of C, again,

$$\mathrm{Colim}_{i \in I} \mathrm{Hom}_{\mathcal{K}'}(C, C) \simeq \mathrm{Hom}_{\mathcal{K}'}(C, \mathrm{Colim}_{i \in I} C)$$

where the latter colimit is that of the diagram in $\mathcal{K}'$. Since $\mathcal{K}'$ is closed under λ^+-directed colimits, $N = \mathrm{Colim}_{i \in I} C$ is in $\mathcal{K}'$, and we define $G(H) = N$. The proof that the compositions of F and G are naturally isomorphic to the identity functors on **C** and **D** is an easy exercise. □

As an aside, for any AEC $\mathcal{K}$ and regular cardinal $\lambda > \mathrm{LS}(\mathcal{K})$, $\mathcal{K}$ is equivalent to the category of presheaves on $\mathcal{K}_{<\lambda}$ that are λ-directed colimits of representable functors (that is, λ-directed colimits of functors of the form $\mathrm{Hom}_{\mathcal{K}}(-, N)$, where N is an object of $\mathcal{K}_{<\lambda}$). The categoricity assumption under which we are currently operating merely guarantees that $\mathcal{K}_{<\lambda^+}$ is a monoid, allowing us to conclude the following:

Theorem 6.5. *Under the hypothesis above, the AEC $\mathcal{K}'$, regarded as a category in the usual way, is equivalent to the category* (M^{op}, λ^+)**-Set**.

Proof. We first note an equivalence of categories between $\mathbf{Set}^{M^{op}}$ and M^{op}**-Set**, in which any functor $H : M^{op} \to \mathbf{Set}$ is sent to the M^{op}-set $(H(C), \rho_H)$, where for any $a \in M$ and $x \in H(C)$, the action is given by $a \cdot x = H(a)(x)$. Explicitly, we have already shown that $\mathcal{K}'$ is equivalent to the full subcategory of $\mathbf{Set}^{M^{op}}$ consisting of all λ^+-directed colimits of $\mathrm{Hom}_{\mathcal{K}'}(-, C)$. Under the new equivalence, $\mathrm{Hom}_{\mathcal{K}'}(-, C)$ maps to the set $\mathrm{Hom}_{\mathcal{K}'}(C, C)$ (that is, M^{op}) with M^{op} acting by precomposition, whereas for arbitrary $N \in \mathcal{K}'$, $\mathrm{Hom}_{\mathcal{K}'}(-, N)$ maps to the set $\mathrm{Hom}_{\mathcal{K}'}(C, N)$, again with M^{op} acting by precomposition. One can see that the image is precisely the full subcategory consisting of λ^+-directed colimits of M^{op} (considered as an M^{op}**-Set**). □

To emphasize, the equivalence between $\mathcal{K}'$ and (M^{op}, λ^+)**-Set** is given by:

$$N \in \mathcal{K}' \quad \mapsto \quad (\mathrm{Hom}_{\mathcal{K}'}(C, N), \rho_N)$$

where the action ρ_N is given, for any $a \in \mathrm{Hom}_{\mathcal{K}'}(C, C)$ and $x \in \mathrm{Hom}_{\mathcal{K}'}(C, N)$, by

$$a \cdot x = x \circ a.$$

A strong embedding $f : N \hookrightarrow_{\mathcal{K}} N'$ is mapped to the M^{op}-set homomorphism $f^* : \mathrm{Hom}_{\mathcal{K}'}(C, N) \to \mathrm{Hom}_{\mathcal{K}'}(C, N')$ that takes any $g \in \mathrm{Hom}_{\mathcal{K}'}(C, N)$ to $f \circ g$. That the map f^* thus defined is in fact a homomorphism of M^{op}-sets is easily verified.

The upshot is this: for any λ-categorical AEC, we may identify $\mathcal{K}' = \mathcal{K}_{\geq\lambda}$ with a category of relatively simple algebraic objects, representing each model by a set equipped with an action of $M^{op} = \mathrm{Hom}_{\mathcal{K}'}(C, C)$, the monoid of endomorphisms of the unique structure in cardinality λ, and replacing the abstract embeddings of $\mathcal{K}$ with concrete homomorphisms between such sets. This gives a radically different context in which to consider questions originally posed in relation to AECs. Conjectures concerned with the upward transfer of categoricity, in particular, involve an analysis of the sub-AEC consisting of the structures whose cardinalities are greater than or equal to the cardinal at which categoricity first occurs; that is, a suitable $\mathcal{K}'$ of the form described above. Given that we have reduced something as complex and general as an AEC to a category whose properties are determined entirely by the structure of the monoid $\mathrm{Hom}_{\mathcal{K}'}(C, C)$ (which is just $\mathrm{Hom}_{\mathcal{K}}(C, C)$, remember), there is some hope that this translation provides a simplification not merely in appearance, but in the sense of providing genuine traction in addressing such problems.

This seems to be one of the strengths of the accessible category viewpoint: it provides new ways of analyzing classes in terms of their smallest structures and the mappings between them.

7. Implications for Galois stability

We now return to the subject broached after Proposition 5.15: in weakly tame and totally transcendental AECs with amalgamation and joint embedding, for certain cardinals λ, weak λ-stability suffices to ensure λ-Galois stability. What makes this interesting is that, thanks to a result of [13], we have, for each AEC $\mathcal{K}$, an infinite list of cardinals λ in which it is weakly λ-stable. For reference, the result in question is:

Proposition 7.1. *Let* $\mathbf{C}$ *be a* λ*-accessible category, and* μ *a regular cardinal such that* $\mu \trianglerighteq \lambda$ *and* $\mu > |\mathbf{Pres}_\lambda(\mathbf{C})^{mor}|$*, where* $\mathbf{Pres}_\lambda(\mathbf{C})^{mor}$ *denotes the set of morphisms in the full subcategory of* $\mathbf{C}$ *consisting of* λ*-presentable objects. Then* $\mathbf{C}$ *is weakly* $\mu^{<\mu}$*-stable.*

We now analyze the import of this proposition in the context of AECs. To simplify the notation, for any AEC $\mathcal{K}$ and cardinal λ we replace the bulky $\mathbf{Pres}_\lambda(\mathcal{K})$ with $\mathcal{A}_{<\lambda}$; that is, we denote by $\mathcal{A}_{<\lambda}$ a full subcategory of $\mathcal{K}$ consisting of one representative of each isomorphism class of models in $\mathcal{K}_{<\lambda}$.

Corollary 7.2. *Let* $\mathcal{K}$ *be an AEC,* $\lambda > LS(\mathcal{K})$ *a regular cardinal, and* μ *a regular cardinal with* $\mu \trianglerighteq \lambda$ *and* $\mu > |(\mathcal{A}_{<\lambda})^{mor}|$*. Then* $\mathcal{K}$ *is weakly* $\mu^{<\mu}$*-stable.*

Proof. By Theorem 4.1, $\mathcal{K}$ is λ-accessible. The result then follows directly from the proposition above. □

We are now finally in a position to apply Theorem 2.11:

Theorem 7.3. *Suppose* $\mathcal{K}$ *satisfies the AP and JEP, is weakly* χ*-tame for some* $\chi \geq LS(\mathcal{K})$*, and is* κ*-totally transcendental with* $\kappa \geq \chi$*. If* $\lambda > LS(\mathcal{K})$ *is a regular cardinal, and* μ *is a regular cardinal with* $\mu > \chi + \kappa$*,* $\mu \trianglerighteq \lambda$*, and* $\mu > |(\mathcal{A}_{<\lambda})^{mor}|$*, then* $\mathcal{K}$ *is* $\mu^{<\mu}$*-Galois stable.*

Proof. By the assumptions on μ, $\mathcal{K}$ is weakly $\mu^{<\mu}$-stable by Corollary 7.2. We show that the conditions of Theorem 2.11 are satisfied, thereby concluding that $\mathcal{K}$ is not merely weakly $\mu^{<\mu}$-stable, but in fact $\mu^{<\mu}$-Galois stable.

Since μ is regular and $\mu > \kappa$, $\mathrm{cf}(\mu^{<\mu}) \geq \mu > \kappa$. Moreover, $\mu^{<\mu} > \chi$. From Proposition 5.15, we know that every $M \in \mathcal{K}_{(\mu^{<\mu})}$ has a Galois-saturated extension $M' \in \mathcal{K}_{(\mu^{<\mu})}$. By the aforementioned theorem, then, we can indeed infer Galois stability in $\mu^{<\mu}$. □

If $\mathcal{K}_{<\lambda}$ contains only a single isomorphism class, say with representative C, $(\mathcal{A}_{<\lambda})^{mor}$ is simply $\mathrm{Hom}_{\mathcal{K}}(C, C)$. This leads to a clearer picture in the following special case:

Proposition 7.4. *If* $\mathcal{K}$ *satisfies the AP and JEP, has* $LS(\mathcal{K}) = \aleph_0$*, and is weakly* $\aleph_0$*-tame, strongly* $\aleph_0$*-categorical, and* $\aleph_0$*-totally transcendental, then for any regular* μ *with* $\aleph_1 \trianglelefteq \mu$ *and* $\mu > |\mathrm{Hom}_{\mathcal{K}}(C, C)|$*,* $\mathcal{K}$ *is* $\mu^{<\mu}$*-Galois stable.*

- Worst case: $|\mathrm{Hom}_{\mathcal{K}}(C, C)| = 2^{\aleph_0}$. We have $(2^{\aleph_1})^+ > |\mathrm{Hom}_{\mathcal{K}}(C, C)|$ and sharply greater than $\aleph_1$, hence also Galois stability in $[(2^{\aleph_1})^+]^{2^{\aleph_1}}$. Similarly, we may infer Galois stability in $[(2^{\aleph_k})^{+(n+1)}]^{(2^{\aleph_k})^{+n}}$ for $1 \leq k \leq \omega$ and $n < \omega$, among other cardinals. Under GCH, this gives Galois stability in all κ with $\aleph_3 \leq \kappa < \aleph_\omega$.
- Better: $|\mathrm{Hom}_{\mathcal{K}}(C, C)| = \aleph_k$ with $0 \leq k < \omega$. Then we have Galois stability in $\aleph_{k+1}^{\aleph_k}$, $\aleph_{k+2}^{\aleph_{k+1}}$, $\aleph_{k+3}^{\aleph_{k+2}}$, and, more generally, $\aleph_{n+1}^{\aleph_n}$ for $k \leq n < \omega$, in addition to the cardinals listed in the worst case scenario above. Under GCH, this gives Galois stability in all κ with $\aleph_{k+1} \leq \kappa < \aleph_\omega$.

One would hope that total transcendence could be replaced by a more straightforward assumption of Galois stability, thereby transforming the above result into a pure upward transfer theorem like those of [3,7,11]. Unfortunately, the proof of the inference from Galois stability to total transcendence hinges on full tameness of the AEC—weak tameness does not suffice. It is to be hoped that a more general argument can be found.

Regardless, we have a partial Galois stability spectrum result (of sorts) for weakly tame AECs and, moreover, the only such result that is not limited to local transfer of the kind covered in [3]. What is most remarkable, perhaps, is the fact that it was derived by largely category-theoretic means, and the way in which it reveals that the proliferation of types over large structures is controlled by the structure of $(\mathcal{A}_{<\lambda})^{mor}$. As in Section 6, this reduction of broad structural questions to ones involving only the smallest models emerges as a central feature – and central virtue – of AECs as seen through the lens of accessible category theory.

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